Math2050A Term1 2017
Tutorial 4, Oct 12

Exercises

1. Let (a_n) be a sequence and $a \in \mathbb{R}$ satisfying the following: for each $\epsilon > 0$, there are infinitely many $n \in \mathbb{N}$ such that $a_n \in (a - \epsilon, a + \epsilon)$.
   Show that there is a subsequence (a_{n_k}) converging to a.

2. Show that a monotone increasing sequence is either convergent or properly diverges to $+\infty$.

3. Suppose (a_n) is a sequence. Show that (a_n) is Cauchy is equivalent to say that $\lim_{n\to\infty} \sup_{p\in\mathbb{N}} |a_{n+p} - a_n| = 0$.

4. Show that every contractive sequence is Cauchy. See textbook[Bartle] **3.5.7 Definition** and **3.5.8 Theorem** in p.88,89.

5. Given $(x_n) \subset \mathbb{R}$. Suppose $\sum_{n=1}^{\infty} |x_n| < \infty$, that is, $\lim_{N\to\infty} \sum_{n=1}^{N} |x_n|$ exists. Show that $\sum_{n=1}^{\infty} x_n$ also exists in $\mathbb{R}$.

6. Given $(x_n) \subset \mathbb{R}$ and $\sum_{n=1}^{\infty} x_n$ exists in $\mathbb{R}$. Show that $\sum_{n=N}^{\infty} x_n \to 0$ as $N \to \infty$.

7. Show the following:

   (a) $\lim\limits_{x\to -1} \frac{x^2+2x+4}{x+2} = 3$.

   (b) $\lim\limits_{x\to 3} \frac{2x+3}{4x-9} = 3$.

   (c) $\lim\limits_{x\to 1} \frac{x^3-1}{x^2-3x+2} = -3$.

Solution

For Q1, Let $n_1 := \min\{n \in \mathbb{N} : a_n \in (a-1, a+1)\}$ and $n_k := \min\{n > n_{k-1} : a_n \in (a - \frac{1}{k}, a + \frac{1}{k})\}$ for $k \geq 2$. By induction, every n_k is well-defined because the set $\{n > n_{k-1} : a_n \in (a-\frac{1}{k}, a+\frac{1}{k})\}$ is nonempty and by well-ordering principle (**1.2.1** in textbook[Bartle] p.12). It is a subsequence because $n_k > n_{k-1}$ for each $k \geq 2$. By squeeze theorem, it converges to a.

For Q6, Let $\epsilon > 0$. Note $(\sum_{k=1}^{n} x_k)_{n=1}^{\infty}$ is a Cauchy sequence in n. There is $N \in \mathbb{N}$ such that for any $m > n \geq N$, we have

$$|\sum_{k=1}^{m} x_k - \sum_{k=1}^{n} x_k| < \epsilon$$

Therefore, $|\sum_{k=n+1}^{m} x_k| < \epsilon$. Letting $m \to \infty$, $|\sum_{k=n+1}^{\infty} x_k| \leq \epsilon$. $|\sum_{k=n}^{\infty} x_k| \leq \epsilon$ holds for any $n \geq N+1$. This completes the proof.

For Q7(a), $|\frac{x^2+2x+4}{x+2} - 3| = |\frac{x^2-x-2}{x+2}| = |\frac{(x+1)(x-2)}{x+2}|$. When $\epsilon > 0$ is given, the proof then is to find a small positive δ such that $|\frac{x^2+2x+4}{x+2} - 3|$ makes sense and $|\frac{x^2+2x+4}{x+2} - 3| < \epsilon$. The δ-neighborhood is the punctured one. We want $x+2$ to be far away from 0. For example, one may restrict $x \in (-\frac{3}{2}, -\frac{1}{2}) \setminus \{-1\}$. In this case, $|\frac{x^2+2x+4}{x+2} - 3| = |\frac{(x+1)(x-2)}{x+2}| \leq |\frac{\frac{7}{2}(x+1)}{\frac{1}{2}}| = 7|x+1|$. If $\delta := \min\{\frac{\epsilon}{7}, \frac{1}{2}\}$, then $(-1-\delta, -1+\delta) \setminus \{-1\} \subset (-\frac{3}{2}, -\frac{1}{2}) \setminus \{-1\}$ and for every $x \in (-1-\delta, -1+\delta) \setminus \{-1\}$, we have $|\frac{x^2+2x+4}{x+2} - 3| \leq 7|x+1| < 7\delta \leq \epsilon$.

For easy grading, it is suggested to have a draft first and then give the proof in a more systematic way as presented in solution for Q7(b),(c).

For Q7(b), Let $\epsilon > 0$.
Let $\delta := \min\{\frac{1}{2}, \frac{\epsilon}{10}\}$. Then, if $x \in (3-\delta, 3+\delta) \setminus \{3\}$, we have

$$|\frac{2x+3}{4x-9} - 3| = |\frac{-10x+30}{4x-9}| = 10|\frac{x-3}{4x-9}| \leq 10|x-3| < 10\delta \leq \epsilon$$

For Q7(c), Let $\epsilon > 0$.
Let $\delta := \min\{\frac{1}{2}, \frac{\epsilon}{14}\}$. Then, if $0 < |x-1| < \delta$, we have

$$|\frac{x^3-1}{x^2-3x+2} + 3| = |\frac{x^3+3x^2-9x+5}{x^2-3x+2}| = |\frac{(x-1)^2(x+5)}{(x-1)(x-2)}| = |\frac{(x-1)(x+5)}{x-2}| \leq 7\frac{|x-1|}{\frac{1}{2}}$$

$$= 14|x-1| < 14\delta \leq \epsilon$$